5.3. Surfaces and their triangulations

In this section, we define (two-dimensional) surfaces, which are topological spaces that locally look like $\mathbb{R}^2$ (and so are supplied with local systems of coordinates). It can be shown that surfaces can always be triangulated (supplied with a PL-structure) We will not prove these two assertions here and limit ourselves to the study of triangulated surfaces (also known as two-dimensional PL-manifolds). The main result is a neat classification theorem, proved by means of some simple piecewise linear techniques and with the help of the Euler characteristic.

5.3.1. Definitions and examples.

DEFINITION 5.3.1. A *closed surface* is a compact connected 2-manifold (without boundary), i.e., a compact connected space each point of which has a neighborhood homeomorphic to the open 2-disk $\operatorname{Int}\mathbb{D}^2$. In the above definition, connectedness can be replaced by path connectedness without loss of generality (see ??)

A *surface with boundary* is a compact space each point of which has a neighborhood homeomorphic to the open 2-disk $\operatorname{Int}\mathbb{D}^2$ or to the open half disk

$$\operatorname{Int}\mathbb{D}^2_{1/2} = \{(x,y) \in \mathbb{R}^2 | x \geqslant 0, x^2 + y^2 < 1\}.$$

EXAMPLE 5.3.2. Familiar surfaces are the 2-sphere $\mathbb{S}^2$, the projective plane $\mathbb{R}P^2$, and the torus $\mathbb{T}^2 = S^1 \times S^1$, while the disk $\mathbb{D}^2$, the annulus, and the Möbius band are examples of surfaces with boundary.

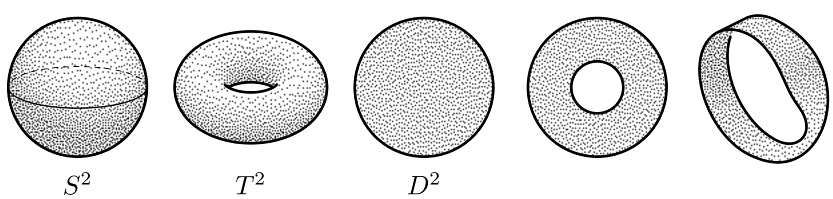

FIGURE 5.3.1. Examples of surfaces

DEFINITION 5.3.3. The *connected sum* $M_1 \# M_2$ of two surfaces M_1 and M_2 is obtained by making two small holes (i.e., removing small open disks) in the surfaces and gluing them along the boundaries of the holes

EXAMPLE 5.3.4. The connected sum of two projective planes $\mathbb{R}P^2 \# \mathbb{R}P^2$ is the famous *Klein bottle*, which can also be obtained by gluing two Möbius bands along their boundaries (see Fig.??). The connected sum of three tori $\mathbb{T}^2 \# \mathbb{T}^2 \# \mathbb{T}^2$ is (topologically) the surface of a pretzel (see Fig.??).

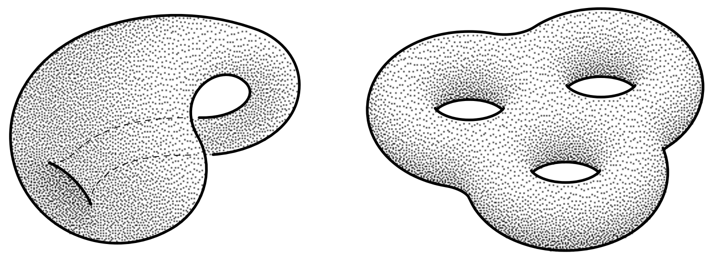

FIGURE 5.3.2. Klein bottle and pretzel

5.3.2. Polyhedra and triangulations. Our present goal is to introduce a combinatorial structure (called PL-structure) on surfaces. First we we give the corresponding definitions related to PL-structures.

A (finite) 2-*polyhedron* is a topological space represented as the (finite) union of triangles (its *faces* or *2-simplices*) so that the intersection of two triangles is either empty, or a common side, or a common vertex. The sides of the triangles are called *edges* or 1-*simplices*, the vertices of the triangles are called *vertices* or 0-*simplices* of the 2-polyhedron.

Let P be a 2-polyhedron and $v \in P$ be a vertex. The (closed) *star* of v in P (notation $\mathrm{Star}(v, P)$) is the set of all triangles with vertex v. The *link* of v in P (notation $\mathrm{Link}(v, P)$) is the set of sides opposite to v in the triangles containing v.

A finite 2-polyhedron is said to be a *closed PL-surface* (or a *closed triangulated surface*) if the star of any vertex v is homeomorphic to the closed 2-disk with v at the center (or, which is the same, if the links of all its vertices are homeomorphic to the circle).

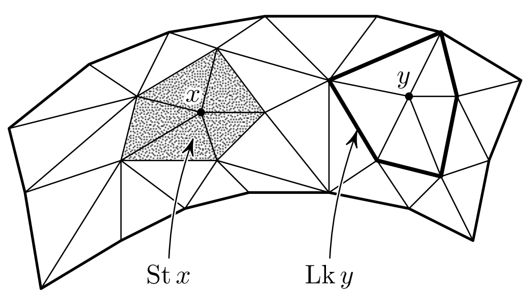

FIGURE 5.3.3. Star and link of a point on a surface

A finite 2-polyhedron is said to be a *PL-surface with boundary* if the star of any vertex v is homeomorphic either to the closed 2-disk with v at the center or to the closed disk with v on the boundary (or, which is the same, if the links of all its vertices are homeomorphic either to the circle or to the line segment). It is easy to see that in a PL-surface with boundary the points whose links are segments (they are called *boundary points*) constitute a finite number of circles (called *boundary circles*). It is also easy to see that each edge of a closed PL-surface (and each nonboundary edge of a surface with boundary) is contained in exactly two faces.

A PL-surface (closed or with boundary) is called *connected* if any two vertices can be joined by a sequence of edges (each edge has a common vertex with the previous one). Further, unless otherwise stated, we consider only connected PL-surfaces.

A PL-surface (closed or with boundary) is called *orientable* if its faces can be coherently oriented; this means that each face can be oriented (i.e., a cyclic order of its vertices chosen) so that each edge inherits opposite orientations from the orientations of the two faces containing this edge. An *orientation* of an orientable surface is a choice of a coherent orientation of its faces; it is easy to see that that any orientable (connected!) surface has exactly two orientations.

A *face subdivision* is the replacement of a face (triangle) by three new faces obtained by joining the baricenter of the triangle with its vertices. An *edge subdivision* is the replacement of the two faces (triangles) containing an edge by four new faces obtained by joining the midpoint of the edge with the two opposite vertices of the two triangles. A *baricentric subdivision* of a face is the replacement of a face (triangle) by six new faces obtained by constructing the three medians of the triangles. A *baricentric subdivision of a surface* is the result of the baricentric subdivision of all its faces. Clearly, any baricentric subdivision can be obtained by means of a finite number of edge and face subdivisions. A *subdivision* of a PL-surface is the result of a finite number of edge and face subdivisions.

Two PL-surfaces M_1 and M_2 are called *isomorphic* if there exists a homeomorphism $h : M_1 \to M_2$ such that each face of M_1 is mapped onto a face of M_2. Two PL-surfaces M_1 and M_2 are called *PL-homeomorphic* if they have isomorphic subdivisions.

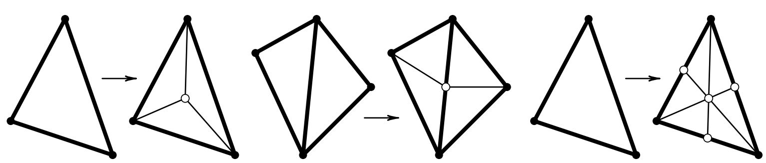

FIGURE 5.3.4. Face, edge, and baricentric subdivisions

EXAMPLE 5.3.5. Consider any convex polyhedron P; subdivide each of its faces into triangles by diagonals and project this radially to a sphere centered in any interior point of P. The result is a triangulation of the sphere.

If P is a tetrahedron the triangulation has four vertices. This is the minimal number of vertices in a triangulation of any surface. In fact, any triangulation of a surface with four vertices is equivalent of the triangulation obtained from a tetrahedron and thus for any surface other than the sphere the minimal number of vertices in a triangulation is greater then four.

EXERCISE 5.3.1. Prove that there exists a triangulation of the projective plane with any given number $N > 4$ of vertices.

EXERCISE 5.3.2. Prove that minimal number of vertices in a triangulation of the torus is six.

5.4. Euler characteristic and genus

In this section we introduce, in an elementary combinatorial way, one of the simplest and most important homological invariants of a surface M – its Euler characteristic $\chi(M)$. The Euler characteristic is an integer (actually defined for a much wider class of objects than surfaces) which is topologically invariant (and, in fact, also homotopy invariant). Therefore, if we find that two surfaces have different Euler characteristics, we can conclude that they are not homeomorphic.

5.4.1. Euler characteristic of polyhedra.

DEFINITION 5.4.1. The *Euler characteristic* $\chi(M)$ of a two-dimensional polyhedron, in particular of a PL-surface, is defined by

$$\boxed{\chi(M) := V - E + F},$$

where V, E, and F are the numbers of vertices, edges, and faces of M, respectively.

PROPOSITION 5.4.2. *The Euler characteristic of a surface does not depend on its triangulation. PL-homeomorphic PL-surfaces have the same Euler characteristic.*

PROOF. It follows from the definitions that we must only prove that the Euler characteristic does not change under subdivision, i.e., under face and edge subdivision. But these two facts are proved by a straightforward verification. □

EXERCISE 5.4.1. Compute the Euler characteristic of the 2-sphere, the 2-disk, the projective plane and the 2-torus.

EXERCISE 5.4.2. Prove that $\chi(M \# N) = \chi(M) + \chi(N) - 2$ for any PL-surfaces M and N. Use this fact to show that adding one handle to an oriented surface decreases its Euler characteristic by 2.

5.4.2. The genus of a surface. Now we will relate the Euler characteristic with a a very visual property of surfaces – their genus (or number of handles). The genus of an oriented surface is defined in the next section (see ??), where it will be proved that the genus g of such a surface determines the surface up to homeomorphism. The model of a surface of genus g is the *sphere with g handles*; for $g = 3$ it is shown on the figure.

PROPOSITION 5.4.3. *For any closed surface M, the genus $g(M)$ and the Euler characteristic $\chi(M)$ are related by the formula*

$$\boxed{\chi(M) \ = \ 2 - 2g(M)}.$$

PROOF. Let us prove the proposition by induction on g. For $g = 0$ (the sphere), we have $\chi(\mathbb{S}^2) = 2$ by Exercise **??**. It remains to show that adding one handle decreases the Euler characteristic by 2. But this follows from Exercise **??** □

REMARK 5.4.4. In fact $\chi = \beta_2 - \beta_1 + \beta_0$, where the β_i are the Betti numbers (defined in **??**). For the surface of genus g, we have $\beta_0 = \beta_2 = 1$ and $\beta_1 = 2g$, so we do get $\chi = 2 - 2g$.

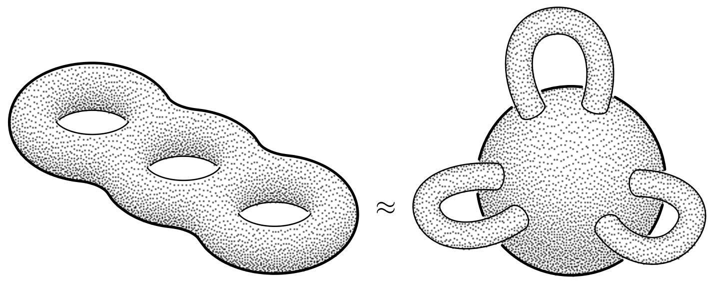

FIGURE 5.4.1. The sphere with three handles

5.5. Classification of surfaces

In this section, we present the topological classification (which coincides with the combinatorial and smooth ones) of surfaces: closed orientable, closed nonorientable, and surfaces with boundary.

5.5.1. Orientable surfaces. The main result of this subsection is the following theorem.

THEOREM 5.5.1 (Classification of orientable surfaces). *Any closed orientable surface is homeomorphic to one of the surfaces in the following list*

$$\mathbb{S}^2,\ \mathbb{S}^1\times\mathbb{S}^1\ (\textit{torus}),\ (\mathbb{S}^1\times\mathbb{S}^1)\#(\mathbb{S}^1\times\mathbb{S}^1)\ (\textit{sphere with 2 handles}),\dots$$

$$\dots,(\mathbb{S}^1\times\mathbb{S}^1)\#(\mathbb{S}^1\times\mathbb{S}^1)\#\dots\#(\mathbb{S}^1\times\mathbb{S}^1)\ (\textit{sphere with k handles}),\dots$$

Any two surfaces in the list are not homeomorphic.

PROOF. By **??** we may assume that M is triangulated and take the double baricentric subdivision M'' of M. In this triangulation, the star of a vertex of M'' is called a *cap*, the union of all faces of M'' intersecting an edge of M but not contained in the caps is called a *strip*, and the connected components of the union of the remaining faces of M'' are called *patches*.

Consider the union of all the edges of M; this union is a graph (denoted G). Let G_0 be a maximal tree of G. Denote by M_0 the union of all caps and strips surrounding G_0. Clearly M_0 is homeomorphic to the 2-disk (why?). If we successively add the strips and patches from $M - M_0$ to M_0, obtaining an increasing sequence

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_p = M,$$

we shall recover M.

Let us see what happens when we go from M_0 to M_1.

If there are no strips left, then there must be a patch (topologically, a disk), which is attached along its boundary to the boundary circle Σ_0 of M_0; the result is a 2-sphere and the theorem is proved.

Suppose there are strips left. At least one of them, say S, is attached along one end to Σ_0 (because M is connected) and its other end is also attached to Σ_0

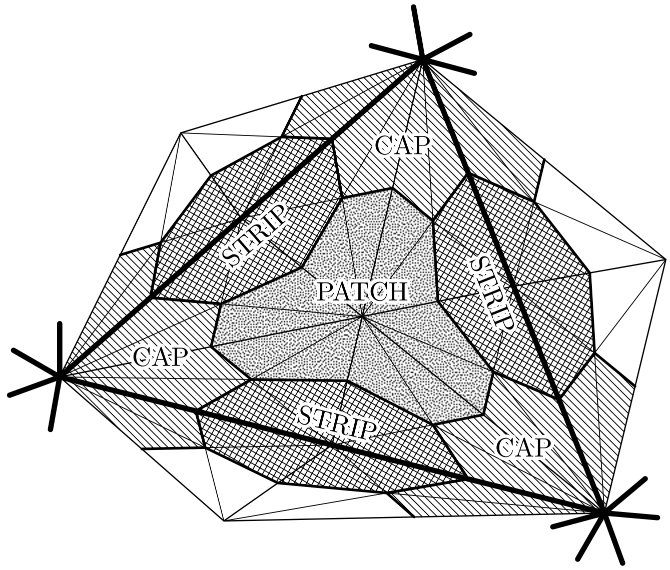

FIGURE 5.5.1. Caps, strips, and patches

(otherwise S would have been part of M_0). Denote by K_0 the closed collar neighborhood of Σ_0 in M_0. The collar K_0 is homoeomorphic to the annulus (and not to the Möbius strip) because M is orientable. Attaching S to M_0 is the same as attaching another copy of $K \cup S$ to M_0 (because the copy of K can be homeomorphically pushed into the collar K). But $K \cup S$ is homeomorphic to the disk with two holes (what we have called "pants"), because S has to be attached in the orientable way in view of the orientability of M (for that reason the twisting of the strip shown on the figure cannot occur). Thus M_1 is obtained from M_0 by attaching the pants $K \cup S$ by the waist, and M_1 has two boundary circles.

FIGURE ??? This cannot happen

Now let us see what happens when we pass from M_1 to M_2.

If there are no strips left, there are two patches that must be attached to the two boundary circles of M_1, and we get the 2-sphere again.

Suppose there are patches left. Pick one, say S, which is attached at one end to one of the boundary circles, say Σ_1 of M_1. Two cases are possible: either

(i) the second end of S is attached to Σ_2, or

(ii) the second end of S is attached to Σ_1.

Consider the first case. Take collar neighborhoods K_1 and K_2 of Σ_1 and Σ_2; both are homoeomorphic to the annulus (because M is orientable). Attaching S to

M_1 is the same as attaching another copy of $K_1 \cup K_2 \cup S$ to M_1 (because the copy of $K_1 \cup K_2$ can be homeomorphically pushed into the collars K_1 and K_2).

FIGURE ??? Adding pants along the legs

But $K - 1 \cup K_2 \cup S$ is obviously homeomorphic to the disk with two holes. Thus, in the case considered, M_2 is obtained from M_1 by attaching pants to M_1 along the legs, thus decreasing the number of boundary circles by one,

The second case is quite similar to adding a strip to M_0 (see above), and results in attaching pants to M_1 along the waist, increasing the number of boundary circles by one.

What happens when we add a strip at the ith step? As we have seen above, two cases are possible: either the number of boundary circles of M_{i-1} increases by one or it decreases by one. We have seen that in the first case "inverted pants" are attached to M_{i-1} and in the second case "upright pants" are added to M_{i-1}.

FIGURE ??? Adding pants along the waist

After we have added all the strips, what will happen when we add the patches? The addition of each patch will "close" a pair of pants either at the "legs" or at the "waist". As the result, we obtain a sphere with k handles, $k \geqslant 0$. This proves the first part of the theorem.

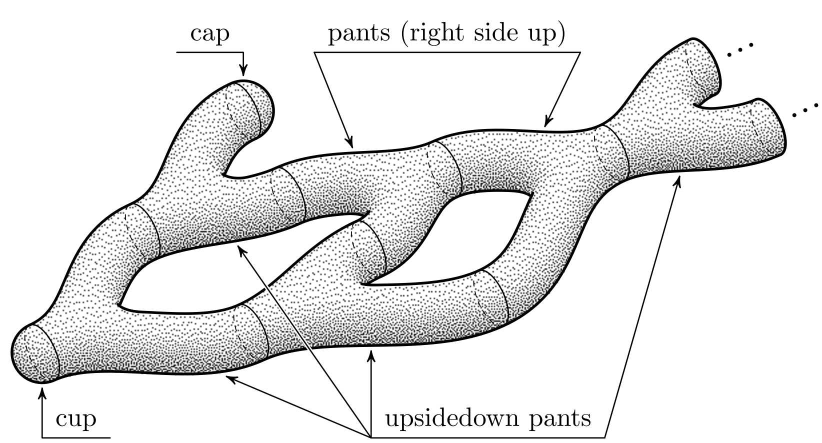

FIGURE 5.5.2. Constructing an orientable surface

To prove the second part, it suffices to compute the Euler characteristic (for some specific triangulation) of each entry in the list of surfaces (obtaining $2, 0, -2, -4, \ldots$, respectively). □

5.5.2. Nonorientable surfaces and surfaces with boundary. Nonorientable surfaces are classified in a similar way. It is useful to begin with the best-known example, the *Möbius strip*, which is the nonorientable surface with boundary obtained by identifying two opposite sides of the unit square $[0, 1] \times [0, 1]$ via $(0, t) \sim (1, 1 - t)$. Its boundary is a circle.

Any compact nonorientable surface is obtained from the sphere by attaching several *Möbius caps*, that is, deleting a disk and identifying the resulting boundary circle with the boundary of a Möbius strip. Attaching m Möbius caps yields a surface of genus $2 - m$. Alternatively one can replace any pair of Möbius caps by a handle, so long as at least one Möbius cap remains, that is, one may start from a sphere and attach one or two Möbius caps and then any number of handles.

All compact surfaces with boundary are obtained by deleting several disks from a closed surface. In general then a sphere with h handles, m Möbius strips, and d deleted disks has Euler characteristic

$$\chi = 2 - 2h - m - d.$$

In particular, here is the finite list of surfaces with nonnegative Euler characteristic:

Surface	h	m	d	χ	Orientable?
Sphere	0	0	0	2	yes
Projective plane	0	1	0	1	no
Disk	0	0	1	1	yes
Torus	1	0	0	0	yes
Klein bottle	0	2	0	0	no
Möbius strip	0	1	1	0	no
Cylinder	0	0	2	0	yes